The analytical value of the corner-ladder graphs contribution to the electron (g-2) in QED.

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Abstract

The contributions to the (g-2) of the electron from the corner-ladder graphs in sixth-order (three-loop) QED perturbation theory are evaluated in closed analytical form. The results obtained are in excellent agreement with the most precise numerical evaluations already existing in the literature. Our results allows one to reduce the numerical uncertainty of the theoretical determination of the g-2 of the electron.

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We have calculated in closed analytical form the contribution to the anomalous magnetic moment of the electron at sixth-order (three-loop) in QED perturbation theory from the so-called corner-ladder graphs shown in Fig.1.

The result are, accounting for the mirror graphs (the infrared terms proportional to $\ln \lambda$ and $\ln^2 \lambda$ are omitted, they are listed in ref.[1]):

$$a(\text{graph } 7) = -\frac{5}{2}\zeta(5) + \frac{5}{9}\pi^2\zeta(3) + \frac{47}{720}\pi^4 + 2\pi^2\ln^2 2 - 28a_4 - \frac{7}{6}\ln^4 2 - \frac{80}{9}\zeta(3)$$
$$+ \frac{55}{18}\pi^2\ln 2 - \frac{2743}{1296}\pi^2 + \frac{2521}{864} \ , \tag{1}$$

$$a(\text{graph } 9) = \frac{95}{24}\zeta(5) - \frac{43}{72}\pi^2\zeta(3) + \frac{199}{2160}\pi^4 + \frac{1}{9}\pi^2\ln^2 2 - \frac{44}{3}a_4 - \frac{11}{18}\ln^4 2$$
$$- \frac{107}{12}\zeta(3) + \frac{25}{18}\pi^2\ln 2 + \frac{3}{16}\pi^2 + \frac{43}{36} \ , \tag{2}$$

$$a(\text{graph } 17) = \frac{25}{6}\zeta(5) - \frac{2}{3}\pi^2\zeta(3) - \frac{31}{540}\pi^4 - \frac{113}{54}\pi^2\ln^2 2 + \frac{200}{9}a_4 + \frac{25}{27}\ln^4 2$$
$$+ \frac{199}{24}\zeta(3) - \frac{155}{18}\pi^2\ln 2 + \frac{3809}{648}\pi^2 - \frac{29}{27} \ , \tag{3}$$

$$a(\text{graph } 19) = \frac{95}{24}\zeta(5) - \frac{3}{8}\pi^2\zeta(3) - \frac{43}{432}\pi^4 + \frac{37}{18}\pi^2\ln^2 2 - \frac{28}{3}a_4 - \frac{7}{18}\ln^4 2$$
$$- \frac{635}{72}\zeta(3) + \frac{83}{18}\pi^2\ln 2 - \frac{4777}{2592}\pi^2 + \frac{1835}{864} \ , \tag{4}$$

$$a(\text{graph } 27) = -\frac{215}{24}\zeta(5) + \frac{95}{72}\pi^2\zeta(3) + \frac{41}{180}\pi^4 - \frac{137}{27}\pi^2\ln^2 2 + \frac{160}{9}a_4 + \frac{20}{27}\ln^4 2$$
$$+ \frac{69}{4}\zeta(3) - \frac{101}{18}\pi^2\ln 2 + \frac{2401}{2592}\pi^2 - \frac{3017}{864} \ , \tag{5}$$

$$a(\text{total}) = \frac{5}{8}\zeta(5) + \frac{17}{72}\pi^2\zeta(3) + \frac{493}{2160}\pi^4 - 3\pi^2\ln^2 2 - 12a_4 - \frac{1}{2}\ln^4 2 - \frac{13}{12}\zeta(3)$$
$$- \frac{31}{6}\pi^2\ln 2 + \frac{655}{216}\pi^2 + \frac{481}{288} \ . \tag{6}$$

Here and in the following $\zeta(p)$ is the Riemann ζ-function of argument p, $\zeta(p) \equiv \sum_{n=1}^{\infty} \frac{1}{n^p}$, (we recall that $\zeta(2) = \pi^2/6$, $\zeta(3) = 1.202\ 056\ 903...$, $\zeta(4) = \pi^4/90$, $\zeta(5) = 1.036\ 927\ 755...$), and $a_4 \equiv \sum_{n=1}^{\infty} \frac{1}{2^n n^4} = 0.517\ 479\ 061...$.

The contributions of these graphs had been previously calculated only by (approximate) numerical methods. The numerical values of eqs.(1)-(6) turn out to be in excellent agreement with the values shown in ref.[1] and [2] (see Table I).

If c_3 is the coefficient of the $\left(\dfrac{\alpha}{\pi}\right)^3$ term in the perturbative expansion of the electron anomaly in QED

$$a_e(\text{QED}) = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) + c_2\left(\frac{\alpha}{\pi}\right)^2 + c_3\left(\frac{\alpha}{\pi}\right)^3 + c_4\left(\frac{\alpha}{\pi}\right)^4 + \dots \; , \tag{7}$$

by using eq.(6) we have

$$c_3 = 1.176\ 19(21) \; , \tag{8}$$

to be compared with the value [3]

$$c_3 = 1.176\ 13(42) \; ; \tag{9}$$

the error of eq.(8) is due to numerical uncertainty of the value of the contribution of the remaining group of five graphs still not known in analytical form [4].
Using the experimental determination of α [5],

$$\alpha^{-1}(\text{exp}) = 137.035\ 997\ 9(32) \; , \tag{10}$$

the values of c_2 and c_4 [6]

$$c_2 = \frac{197}{144} + \frac{1}{2}\zeta(2) + \frac{3}{4}\zeta(3) - 3\zeta(2)\ln 2 = -0.328\ 478\ 965...$$

$$c_4 = -1.434(138) \tag{11}$$

and accounting for the small vacuum polarization contributions of muon, tau and hadron loops as well as the electroweak contribution [6] one finds

$$a_e(\text{theory}) = 1\ 159\ 652\ 141.4(27.1)(2.6)(4.1) \times 10^{-12} \tag{12}$$

where the first error comes from the error of the determination of α (10), the second from the value of c_3 and the third from the error of c_4. The error due to the three-loop coefficient c_3 is now less than the error due to c_4.

If one assumes the validity of QED, using the experimental value of the electron anomaly [7]

$$a_e(\text{exp}) = 1\ 159\ 652\ 188.4(4.3) \times 10^{-12} \; , \tag{13}$$

one can estimate α from eq.(7); one finds

$$\alpha^{-1}(a_e) = 137.035\ 992\ 34(51)(31)(48) \; , \tag{14}$$

where the errors come respectively, from eq.(13), from c_3 and from c_4. Summing up the errors of eq.(14) in quadrature one finds

$$\alpha^{-1}(a_e) = 137.035\ 992\ 34(77) \tag{15}$$

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which is the most accurate determination of α available at present.

We describe now briefly the techniques used for deriving eqs.(1)-(6). All the vertex graphs of Fig.1 can be obtained from the self-mass–like graph of Fig.2 by inserting the external photon line in all possible ways along the electron line. Once that the limit $\Delta \to 0$ is taken, the contribution from each of graphs of Fig.1 is expressed as a sum of some hundreds of terms all of the form N/D, where N is a scalar polynomial in p and the internal integration momenta, and $D = D_1^{n_1} D_2^{n_2} \ldots D_8^{n_8} \ (n_i = 0, 1, 2),$ the D_i being the denominators of the various propagators of the graph of Fig.2. The simplest term, with $N = 1$ and all $n_i = 1$, was calculated in analytical form in a precedent paper [8].

The contributions of graphs 19 and 27 turn out to be finite; on the contrary, graphs 7, 9 and 17 need renormalization, and as consequence their contributions become infrared divergent; in this case appropriate ultraviolet counterterms are subtracted, as well as infrared ones which render the entire contributions finite.

Let us consider for instance the term defined as

$$I = \frac{1}{\pi^6} \int (-i)^3 d^4q \ d^4k_1 \ d^4k_2 \ \frac{N}{D} \ , \tag{16}$$

with $N = (k_1 - k_2)^2$ and

$$D = (k_1^2 - i\epsilon)(k_2^2 + 1 - i\epsilon)((k_1 - k_2)^2 + 1 - i\epsilon)((p - k_1)^2 + 1 - i\epsilon)((p - k_2)^2 - i\epsilon) \times$$

$$\times (q^2 + 1 - i\epsilon)((q - k_1)^2 + 1 - i\epsilon)((q - k_2)^2 - i\epsilon)$$

(the electron mass is set equal to 1). In order to evaluate eq.(16) we apply the techniques used for calculating the analytical contribution to the g-2 of the muon of the "light-light" vertex graphs [9]. The self-mass–like graph of Fig.2 is depicted so as to show the topological similarity with the correspondent self-mass–like graph obtained from the "light-light" graphs. Following [9], we write a dispersion relation in $(k_1 - k_2)^2$ at constant $k_1^2 = l$, $k_2^2 = m$, for the inserted q-loop vertex part

$$\int \frac{(-i)d^4q}{(q^2 + 1 - i\epsilon)((q - k_1)^2 + 1 - i\epsilon)((q - k_2)^2 - i\epsilon)} =$$

$$= \pi^2 \int_1^\infty \frac{db}{b + (k_1 - k_2)^2} \frac{1}{R(b, -l, -m)} \ln W \ ,$$

where
$$\ln W \equiv \ln \frac{b(b + l + m) + (b - l + m) + (b - 1) \ R(b, -l, -m)}{b(b + l + m) + (b - l + m) - (b - 1) \ R(b, -l, -m)} \ , \tag{18}$$

and $R(x, y, z)$ is the usual two-body phase space square root

$$R(x, y, z) \equiv \sqrt{(x - y - z)^2 - 4yz} = \sqrt{x^2 + y^2 + z^2 - 2xy - 2yz - 2xz} \ . \tag{19}$$

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Now we introduce hyperspherical variables for both the k_1 and the k_2 loops

$$d^4 k_1 = \frac{1}{2} l \ dl \ d\Omega_4(\hat{k}_1) \ ,$$

$$d^4 k_2 = \frac{1}{2} m \ dm \ d\Omega_4(\hat{k}_2) \ ,$$

(20)

and perform the hyperspherical angular integrals by expanding the denominators in Gegenbauer polynomials [10]. In the mass-shell limit $p^2 = -1$ we obtain

$$\int \frac{d\Omega_4(\hat{k}_1) d\Omega_4(\hat{k}_2)}{((p - k_1)^2 + 1)(p - k_2)^2 (b + (k_1 - k_2)^2)} = \frac{\left(2\pi^2\right)^2}{lm} \ln Y_{\pm} \ ,$$

(21)

where

$$Y_{\pm} \equiv 1 + \frac{1}{8ml} \left[\sqrt{l(l+4)} - l \right] \left[(m+1) \pm (m-1)\right] \left[b + l + m - \ R(b, -l, -m)\right] \ .$$

The analytical continuation for timelike values of p^2 of the radial integral over m requires also a deformation of the contour [10] of the integration in order to avoid the singularity at $m = -1$. The contour is split in two parts, one where $m \geq -1$ and another where $-1 \leq m \leq 0$. Y_- must be used when integrating over m in the first zone and and Y_+ in the second one. At this point the integral (16) reads

$$I = \int_0^{\infty} \frac{dl}{l} \left(\int_{-1}^{\infty} \frac{dm}{m+1} \int_1^{\infty} \frac{db}{R(b, -l, -m)} \ln W \ln Y_- \right.$$

$$\left. - \int_{-1}^0 \frac{dm}{m+1} \int_1^{\infty} \frac{db}{R(b, -l, -m)} \ln W \ln Y_+ \right) \ .$$

(22)

Eq.(22) contains two square roots: $R(b, -l, -m)$ and $\sqrt{l(l+4)}$. We choose to integrate eq.(22) in b first, then in m and at last in l; we used this order of integration for all the other integrals over b, m and l needed in the calculation of eqs.(1)-(6) [1]. If we write

$$I = \int_0^{\infty} \frac{dl}{l} f(l) \ ,$$

the b and m integrals are performed by differentiating (repeatedly, when needed) with respect to l (see ref.[8]), then integrating the resulting (simpler) expression, and re-covering $f(l)$ by a quadrature. Note that the derivatives with respect to l, m and

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[1]   Note the simplification with respect to the analogous "light-light" graphs calculation; in eq.(23) of ref.[9] four square roots appear, and as consequence the decomposition of the hyperspherical logarithm is needed and different orders of integration must be chosen for the different pieces.

b of the functions $\ln W$ and $\ln Y_\pm$ contain the same polynomial in the denominator, $(b+m)^2 + l(b-1)(m+1)$, even if the two functions have different origin; this fact simplifies considerably the calculations. At last, $f(l)$ is expressed as a sum of known tetralogarithms of the variable $x = (l + 2 - \sqrt{l(l+4)})/2$:

$$
\begin{aligned}
f(l) = & - 9\mathrm{Li}_4(x) - 15\mathrm{Li}_4(-x) + \frac{1}{2}S_{2,2}(x^2) - 2S_{2,2}(-x) + 2T_{2,2}(x) - \mathrm{Li}_2^2(x) \\
& + \ln x \left(5\mathrm{Li}_3(x) + 8\mathrm{Li}_3(-x) - 2S_{1,2}(x) - 2\mathrm{Li}_2(x)\ln(1-x)\right) \\
& - \ln^2 x \left(\mathrm{Li}_2(x) + \frac{3}{2}\mathrm{Li}_2(-x) + \frac{1}{2}\ln^2(1-x)\right) + 2\zeta(2)\mathrm{Li}_2(-x) ,
\end{aligned}
\tag{23}
$$

where

$$
S_{n,p}(x) \equiv \frac{(-1)^{n+p-1}}{(n-1)!\,p!}\int_0^1 \frac{dt}{t}\ln^{n-1}(t)\ln^p(1-xt) ,
$$

$$
\mathrm{Li}_{n+1}(x) \equiv S_{n,1}(x) ,
$$

and [2]

$$
T_{2,2}(x) \equiv \int_0^x \frac{dt}{t}\ln(t+1)\mathrm{Li}_2(t) .
$$

The last definite integral over l is performed using known tables of integrals containing polylogarithms; one finds

$$
I = \frac{21}{2}\zeta(2)\zeta(3) - \frac{45}{4}\zeta(5) .
\tag{24}
$$

The processing of the other terms with different N and D is similar to that described above. When the numerator N contains powers of q we use the decomposition described in ref.[9]; as this fact causes a severe blowing-up of the size of the calculations, some algebraic manipulations are used in order to reduce the maximum power to q^3. In the course of the calculations, some non-standard functions of polylogarithmic degree 4 appear, as an example

$$
g_1(x) = \int_0^x dt\,\ln(1-t+t^2)\frac{\mathrm{Li}_2(t)}{t} ;
$$

the expressions of the functions $g_i(x)$ by means of polylogarithmic functions $S_{n,p}(x)$ are not known. The last integration over l of the functions $g_i(x)$, after some integration by parts, give a group of definite integrals containing $\ln(1-t+t^2)$ which we found to be the same ones appeared in the calculation of the contribution to the g-2 of the electron

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[2] The expression of the function $T_{2,2}(x)$ in terms of polylogarithmic functions $S_{n,p}(x)$ is not known; nevertheless, the properties of $T_{2,2}(x)$ are similar to those of $S_{n,p}(x)$, so that we consider this function as a "standard" polylogarithm.

of another set of vertex graphs [11], and which were calculated in analytical form in ref.[12].

The final infrared-cutoff dependent results (1)-(6) are obtained using the analytical expressions of the infrared counterterms shown in ref.[13].

As a check, for each graph of Fig.1 we have modified the internal routing of the momentum Δ and we have extracted the contribution to the $(g\text{-}2)$, obtaining new expressions, whose subsequent analytical evaluations reproduce correctly eqs.(1)-(6).

The algebra of the whole calculation was processed relying on the algebraic manipulation program ASHMEDAI [14]. All calculations were done on a number of VAXstation 4000/60 and 4000/90. The processing of the whole contribution of a graph required about 40h of continued CPU time and 150 Mbytes of disk space for storing of intermediate expressions on a VAXstation 4000/90, the peak length of the algebraic expressions being about half million of terms. Due to the (relative) high speed of machines used, we devoted our efforts to design and carry out the maximum quantity of consistency and cross checks, rather than to speed up the execution time or to minimize the disk space requirements.

Work at the contribution to the electron anomaly of the remaining graphs still not known in analytical form is now in progress.

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Figure captions

Fig.1: The corner-ladder graphs (mirror graphs are omitted). The numbering follows ref.[1].

Fig.2: The self-mass–like scalar graph with the cut of the dispersion relation.

Table I

Graph	ref.[1]	ref.[2]	Our
7	$-2.6707(19)$	$-2.670\,546(30)$	$-2.670\,554\,745\,651...$
9	$0.6189(64)$	$0.617\,727(121)$	$0.617\,711\,782\,178...$
17	$0.6097(34)$	$0.607\,660(240)$	$0.607\,752\,806\,792...$
19	$-0.3182(72)$	$-0.334\,698(11)$	$-0.334\,695\,103\,723...$
27	$1.8572(86)$	$1.861\,992(240)$	$1.861\,907\,872\,591...$
Total	$0.0893(60)$	$0.082\,065(362)$	$0.082\,122\,612\,187...$

Table I: Comparison of our values of the contributions of the graphs shown in Fig.1 with the values of ref.[1] and ref.[2].

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